

A regularized penalty-multiplier method for approximating cavitation solutions with prescribed cavity volume size

Pablo V. Negrón–Marrero
 Department of Mathematics
 University of Puerto Rico
 Humacao, PR 00791-4300
 pablo.negron1@upr.edu

Jeyabal Sivaloganathan
 Department of Mathematical Sciences
 University of Bath, Bath
 BA2 7AY, UK
 masjs@bath.ac.uk

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Abstract

Let $\Omega \in \mathbb{R}^n$ be the region occupied by a body and let \mathbf{x}_0 be a flaw point in Ω . Let $E(\cdot)$ be an energy functional (defined on some appropriate admissible set of deformations of Ω). For $V > 0$ fixed, we let \mathbf{u}_V be a minimizer of $E(\cdot)$ among the set of deformations constrained to form a hole of volume V at \mathbf{x}_0 . In this paper we describe a regularized penalty–multiplier method and its convergence properties for the computation of both \mathbf{u}_V and $E(\mathbf{u}_V)$. In particular, we show that as the regularization parameter goes to zero, the regularized constrained minimizers converge weakly in $W^{1,p}(\Omega \setminus \overline{\mathcal{B}_\delta(\mathbf{x}_0)})$ to \mathbf{u}_V for any $\delta > 0$. We describe as well the main features of a numerical scheme for approximating \mathbf{u}_V and $E(\mathbf{u}_V)$ and give a numerical example for the case of a stored energy for an elastic fluid.

1 Introduction

When certain materials, such as rubber, are subjected to sufficiently large triaxial loading, holes or bubbles begin to appear inside of the stressed specimen (see, e.g., Gent and Lindley [4], Gent [3]). The first variational model, based on the equations of nonlinear elasticity, that predicts this phenomenon of void formation was given by Ball in [2]. In

this paper, Ball modelled a spherical body composed of an isotropic “soft” material and showed that, in the class of radial deformations, any minimiser of the stored energy functional must open a hole at the centre of the deformed ball for sufficiently large boundary displacements (the phenomenon of cavitation). After this seminal paper, many others appeared on different aspects of radial cavitation, e.g., [1], [9], [10], [11], [17], [20], among others. For results on general nonsymmetric cavitation in elasticity we refer to [12], [14], [6] and [18]. A fundamental problem in studies of cavitation is to mathematically or computationally characterize the boundary displacements for which cavitation occurs. In [15] the authors introduced the concept of the *volume derivative* as a tool for characterizing these boundary displacements. For a large class of materials, the onset of cavitation-type instabilities can be characterized as the zero level set of the volume derivative.

Central to any scheme for approximating the volume derivative is the computation of minimizers of the corresponding stored energy functional satisfying the constraint of forming or developing a hole inside the body of a prescribed volume. This is the problem that concerns us in this paper, in particular we describe a regularized penalty–multiplier method and its convergence properties for the computation of such minimizers. This question was partially addressed in [15] where a penalty method on a punctured domain is discussed. The convergence of this method as the penalty parameter increased to infinity was established, but no result was given as the radius of the spherical “regularizing” incision goes to zero. We provide such a result in this paper. Moreover, the use of a penalty–multiplier technique leads to a more stable numerical scheme as compared to a standard penalty method. For more details on the penalty–multiplier method, also called *augmented Lagrangians*, we refer to [13] and the references therein.

This paper is similar in spirit to [19] in which a regularization method for cavitating solutions is described. However the problem considered in [19] is without the constraint that the deformation opens a hole inside the body of a prescribed volume. The presence of this nonlinear volume constraint leads to various technical difficulties among which is that of constructing variations satisfying this constraint. The use of the penalty–multiplier technique eliminates the need for constructing such variations by replacing the original constrained problem with a sequence of unconstrained problems.

To introduce the results in the paper, consider a nonlinear hyperelastic body occupying the bounded region $\Omega \subset \mathbb{R}^n$ in its reference state. A deformation of the body is a mapping $\mathbf{u} : \Omega \rightarrow \mathbb{R}^n$ satisfying the local invertibility condition

$$\det \nabla \mathbf{u}(\mathbf{x}) > 0 \quad \text{a.e. } \mathbf{x} \in \Omega. \quad (1.1)$$

The energy stored in the deformed body under a deformation \mathbf{u} is given by

$$E(\mathbf{u}) = \int_{\Omega} W(\nabla \mathbf{u}(\mathbf{x})) \, d\mathbf{x}, \quad (1.2)$$

where $W : M_+^{n \times n} \rightarrow \mathbb{R}$ is the stored energy function of the material and $M_+^{n \times n}$ denotes the set of $n \times n$ matrices with positive determinant. In this paper we consider the *displacement boundary value problem* in which we fix a matrix $\mathbf{A} \in M_+^{n \times n}$ and consider deformations satisfying

$$\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x} \text{ for } \mathbf{x} \in \partial\Omega, \quad (1.3)$$

We next fix a “flaw” point $\mathbf{x}_0 \in \Omega$, and for any fixed $V > 0$ we take the *admissible set* of deformations to be

$$\begin{aligned} \mathcal{A}_{\mathbf{A},V} = \{ & \mathbf{u} \in W^{1,p}(\Omega) \mid \text{Det} \nabla \mathbf{u} = \det \nabla \mathbf{u} \, \mathcal{L}^n + V \delta_{\mathbf{x}_0}, \det \nabla \mathbf{u} > 0 \text{ a.e.,} \\ & \mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x} \text{ on } \partial\Omega, \mathbf{u} \text{ satisfies INV on } \Omega \}. \end{aligned} \quad (1.4)$$

Here $\text{Det} \nabla \mathbf{u}$ denotes the distributional determinant of \mathbf{u} , defined by

$$\langle \text{Det} \nabla \mathbf{u}, \phi \rangle = -\frac{1}{n} \int_{\Omega} \nabla \phi \cdot (\text{adj } \nabla \mathbf{u}) \mathbf{u} \, d\mathbf{x}, \quad \forall \phi \in C_0^\infty(\Omega), \quad (1.5)$$

\mathcal{L}^n denotes n -dimensional Lebesgue measure, $p > n - 1$, $\delta_{\mathbf{x}_0}$ denotes the Dirac measure supported at $\mathbf{x}_0 \in \Omega$, and (INV) denotes the condition, relating to invertibility, introduced in Definition 3.2 of [14]. Results in [18] give conditions on the stored energy function W under which a minimiser for (1.2) exists on the set $\mathcal{A}_{\mathbf{A},V}$. The results of Henao and Mora-Corral [6] give conditions under which a minimiser also exists in the case $p = n - 1$ and their work in [7] includes justification of the interpretation of V in (1.4) as the volume of the hole formed by the deformation. Hence, if $\mathbf{u} \in \mathcal{A}_{\mathbf{A},V}$, then the deformation \mathbf{u} produces a hole of volume V in the deformed body.

In Appendix A we show that the requirement on deformations of producing a hole of volume V in the deformed body is equivalent to the following *integral constraint*:

$$\int_{\Omega} \det \nabla \mathbf{u} \, d\mathbf{x} = (\det \mathbf{A}) |\Omega| - V.$$

(Here $|\Omega|$ is the volume of Ω .) Thus we replace the minimization of (1.2) over (1.4) with

$$\begin{cases} \inf_{\mathbf{u} \in \mathcal{A}_{\mathbf{A}}} & E(\mathbf{u}), \\ \text{subject to} & c(\mathbf{u}) = 0, \end{cases} \quad (1.6)$$

where now

$$\mathcal{A}_{\mathbf{A}} = \left\{ \mathbf{u} \in W^{1,p}(\Omega) \mid \exists \alpha \geq 0 \text{ such that } \text{Det} \nabla \mathbf{u} = \det \nabla \mathbf{u} \mathcal{L}^n + \alpha \delta_{\mathbf{x}_0}, \right. \\ \left. \det \nabla \mathbf{u} > 0 \text{ a.e., } \mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x} \text{ on } \partial\Omega, \mathbf{u} \text{ satisfies INV on } \Omega \right\},$$

and

$$c(\mathbf{u}) = \int_{\Omega} \det \nabla \mathbf{u} \, d\mathbf{x} - (\det \mathbf{A}) |\Omega| + V. \quad (1.7)$$

For any $\varepsilon > 0$, let

$$\Omega_{\varepsilon} = \Omega \setminus \overline{\mathcal{B}_{\varepsilon}(\mathbf{x}_0)}.$$

(Here and henceforth, we use the notation $\mathcal{B}_{\varepsilon}(\mathbf{x}_0)$ for the open ball of radius ε centered at \mathbf{x}_0 .) The *regularized constrained* minimization problem is given by:

$$\begin{cases} \inf_{\mathbf{u} \in \mathcal{A}_{\mathbf{A}}^{\varepsilon}} & E_{\varepsilon}(\mathbf{u}), \\ \text{subject to} & c_{\varepsilon}(\mathbf{u}) = 0. \end{cases} \quad (1.8)$$

where

$$E_{\varepsilon}(\mathbf{u}) = \int_{\Omega_{\varepsilon}} W(\nabla \mathbf{u}(\mathbf{x})) \, d\mathbf{x}, \quad (1.9)$$

$$c_{\varepsilon}(\mathbf{u}) = \int_{\Omega_{\varepsilon}} \det \nabla \mathbf{u} \, d\mathbf{x} - (\det \mathbf{A}) |\Omega| + V, \quad (1.10)$$

and

$$\mathcal{A}_{\mathbf{A}}^{\varepsilon} = \left\{ \mathbf{u} \in W^{1,p}(\Omega_{\varepsilon}) \mid \text{Det} \nabla \mathbf{u} = (\det \nabla \mathbf{u}) \mathcal{L}^n, \det \nabla \mathbf{u} > 0 \text{ a.e.,} \right. \\ \left. \mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x} \text{ on } \partial\Omega, \mathbf{u} \text{ satisfies INV} \right\},$$

We set $\mathcal{A}_{\mathbf{A}}^0 = \mathcal{A}_{\mathbf{A}}$ and $c_0 = c$. In Appendix B we show that the admissible sets

$$\mathcal{C}_{\mathbf{A}}^{\varepsilon} \equiv \{ \mathbf{u} \in \mathcal{A}_{\mathbf{A}}^{\varepsilon} \mid c_{\varepsilon}(\mathbf{u}) = 0 \},$$

are nonempty for ε sufficiently small.

Remark 1.1. The hypotheses and results of [18] are easily adapted to prove that a (not necessarily unique) minimiser $\mathbf{u}_{V,\varepsilon}$ of E_{ε} on $\mathcal{A}_{\mathbf{A}}^{\varepsilon}$ exists for each $\varepsilon \geq 0$ and $V > 0$ small enough.

To compute approximations of the constrained problem (1.8), we use a penalty–multiplier method in which the functional in (1.9) is replaced with:

$$E_{\varepsilon,\mu,\eta}(\mathbf{u}) = E_{\varepsilon}(\mathbf{u}) + \mu c_{\varepsilon}(\mathbf{u}) + \frac{1}{2}\eta c_{\varepsilon}(\mathbf{u})^2. \quad (1.11)$$

Here η is a “large” positive parameter and $\mu \in \mathbb{R}$. Thus we replace the constrained problem (1.8) with the “unconstrained” problem:

$$\inf_{\mathbf{u} \in \mathcal{A}_{\mathbf{A}}^{\varepsilon}} E_{\varepsilon,\mu,\eta}(\mathbf{u}). \quad (1.12)$$

In Proposition 2.1 we show that a minimizer $\mathbf{u}_{V,\varepsilon,\mu,\eta}$ of (1.12) actually exists. This result is then used in Theorem 2.2 to show that for fixed $\varepsilon, V > 0$, there exist sequences $\{\mu_j\}$, $\{\eta_j\}$ such that $\{\mathbf{u}_{V,\varepsilon,\mu_j,\eta_j}\}$ converges weakly in $W^{1,p}(\Omega_{\varepsilon})$ to a solution $\mathbf{u}_{V,\varepsilon}$ of (1.8), and $c_{\varepsilon}(\mathbf{u}_{V,\varepsilon,\mu_j,\eta_j}) \rightarrow 0$. We conclude Section 2 with a result on the weak form of the Euler–Lagrange equations for the minimizer $\mathbf{u}_{V,\varepsilon}$ (Theorem 2.3) and a result on the sensitivity of the minimum energy $E_{\varepsilon}(\mathbf{u}_{V,\varepsilon})$ to variations on the boundary data \mathbf{A} (Theorem 2.4).

In Section 3 we prove the main result of this paper, namely Theorem 3.2. We show that for a sequence $\{\varepsilon_j\}$ converging to zero, the regularized constrained minimizers $\{\mathbf{u}_{V,\varepsilon_j}\}$ converge weakly in $W^{1,p}(\Omega_{\delta})$ to a solution \mathbf{u}_V of (1.6), for any $\delta > 0$. The first part of the proof of this result, dealing with the convergence and the existence of the limit \mathbf{u}_V , is very similar to that in [19, Theorem 4.1] and uses a “diagonalization” argument. However showing that the limiting function \mathbf{u}_V is actually a solution of (1.6) is more subtle, again due to the presence of the integral volume constraint in (1.6). A key ingredient to get this result is Lemma 3.1 which shows that the energy of any function $\mathbf{u} \in \mathcal{C}_{\mathbf{A}}^0$ can be arbitrarily approximated by the energies of functions in $\mathcal{C}_{\mathbf{A}}^{\varepsilon}$. The proof of this lemma involves the construction of certain diffeomorphisms of regularized sets where we needed to assume the convexity of Ω . We close Section 3 with a result on the weak form of the Euler–Lagrange equations for the minimizer \mathbf{u}_V (Theorem 3.3).

Finally in Section 4 we describe the main features of a numerical scheme for approximating solutions of the problem (1.6). Also we give a numerical example for the case of an elastic fluid ($\mu = 0$ in (1.13)). For this class of materials and for a spherical domain, an exact solution of (1.6) is known and thus we can check the various convergence results in the paper in this case.

A simple class of polyconvex isotropic stored energy functions to which the results in this paper can be applied is given by

$$W(\mathbf{F}) = \frac{\mu}{q} \|\mathbf{F}\|^q + h(\det \mathbf{F}), \quad (1.13)$$

where $\kappa > 0$, $q \in [n-1, n)$ and $h : (0, \infty) \rightarrow (0, \infty)$ is such that

$$h \text{ is a } C^2, \text{ convex function and} \quad (1.14a)$$

$$h(\delta) \rightarrow \infty \text{ and } \frac{h(\delta)}{\delta} \rightarrow \infty \text{ as } \delta \rightarrow 0, \infty \text{ respectively.} \quad (1.14b)$$

However, we note that the results of this paper apply to more general polyconvex stored energy functions under varied hypotheses.

2 The regularized penalty–multiplier method

In this section we study the unconstrained problem (1.12), in particular the convergence properties of the penalty–multiplier method. We assume that the stored energy function $W(\mathbf{F})$ satisfy the following:

H1: (Polyconvexity) There exists with $G : (M_+^{n \times n})^{n-1} \times (0, \infty) \rightarrow \mathbb{R}$ continuous and convex such that

$$W(\mathbf{F}) = \begin{cases} G(\mathbf{F}, \det \mathbf{F}) & , \quad n = 2, \\ G(\mathbf{F}, \text{adj } \mathbf{F}, \det \mathbf{F}) & , \quad n = 3. \end{cases}$$

H2: (Growth) For $p \in (n-1, n)$, $c_1 > 0$, and a C^2 function h , we have that

$$W(\mathbf{F}) \geq K + c_1 |\mathbf{F}|^p + h(\det \mathbf{F}) \quad \text{for } \mathbf{F} \in M_+^{n \times n},$$

where the function h satisfies conditions (1.14).

We begin by showing that the minimizers in (1.12) actually exists.

Proposition 2.1. *For any $\mu \in \mathbb{R}$, $\eta > 0$, the infimum*

$$\inf_{\mathbf{u} \in \mathcal{A}_{\mathbf{A}}^\varepsilon} E_{\varepsilon, \mu, \eta}(\mathbf{u}),$$

exists and is attained for a function $\mathbf{u}_{V, \varepsilon, \mu, \eta} \in \mathcal{A}_{\mathbf{A}}^\varepsilon$. Moreover, for any $\delta > 0$, the parameter η can be chosen sufficiently large such that the minimizer $\mathbf{u}_{V, \varepsilon, \mu, \eta}$ satisfies that $|c_\varepsilon(\mathbf{u}_{V, \varepsilon, \mu, \eta})| < \delta$.

Proof: Since $\mathcal{A}_{\mathbf{A}}^\varepsilon \neq \emptyset$, the infimum above is less than ∞ . If the infimum were $-\infty$, there would exists a sequence $\{\mathbf{u}_k\}$ in $\mathcal{A}_{\mathbf{A}}^\varepsilon$ such that $E_{\varepsilon, \mu, \eta}(\mathbf{u}_k) \rightarrow -\infty$ as $k \rightarrow \infty$, i.e.,

$$\int_{\Omega_\varepsilon} W(\nabla \mathbf{u}_k(\mathbf{x})) \, d\mathbf{x} + \mu c_\varepsilon(\mathbf{u}_k) + \frac{1}{2} \eta c_\varepsilon(\mathbf{u}_k)^2 \rightarrow -\infty, \quad k \rightarrow \infty.$$

Since the first and third terms above are bounded below, and for $\mathbf{u}_k \in \mathcal{A}_{\mathbf{A}}^\varepsilon$ we have that $c_\varepsilon(\mathbf{u}_k)$ is bounded below as well, it follows that if $\mu \geq 0$, the above limit can not be $-\infty$. If $\mu < 0$, for the above limit to be $-\infty$, we would need to have (for a subsequence) that $c_\varepsilon(\mathbf{u}_k) \rightarrow \infty$. In this last case:

$$\begin{aligned} \int_{\Omega_\varepsilon} W(\nabla \mathbf{u}_k(\mathbf{x})) \, d\mathbf{x} + \mu c_\varepsilon(\mathbf{u}_k) + \frac{1}{2} \eta c_\varepsilon(\mathbf{u}_k)^2 &\geq c_\varepsilon(\mathbf{u}_k) \left(\mu + \frac{1}{2} \eta c_\varepsilon(\mathbf{u}_k) \right), \\ &\rightarrow \infty, \quad \text{as } k \rightarrow \infty, \end{aligned}$$

where we used that $\eta > 0$. Thus in any case we arrive at a contradiction and we must have that

$$\inf_{\mathbf{u} \in \mathcal{A}_{\mathbf{A}}^\varepsilon} E_{\varepsilon, \mu, \eta}(\mathbf{u}) = g^* \in \mathbb{R}.$$

Let now $\{\mathbf{u}_k\}$ in $\mathcal{A}_{\mathbf{A}}^\varepsilon$ be an infimizing sequence, i.e., $E_{\varepsilon, \mu, \eta}(\mathbf{u}_k) \rightarrow g^*$. An argument similar to the one above implies that $\{c_\varepsilon(\mathbf{u}_k)\}$ must be bounded. If $\mu c_\varepsilon(\mathbf{u}_k) \geq -L$ for all k , where $L > 0$, then for k sufficiently large we get that

$$\int_{\Omega_\varepsilon} W(\nabla \mathbf{u}_k(\mathbf{x})) \, d\mathbf{x} - L \leq g^* + 1.$$

It follows now from the growth hypotheses (H1)–(H2) that there exists a subsequence $\{\mathbf{u}_{k_j}\}$ which converges weakly in $W^{1,p}(\Omega_\varepsilon)$ to a function \mathbf{u}^* , and that $\{\det \nabla \mathbf{u}_{k_j}\}$ converges weakly in $L^1(\Omega_\varepsilon)$ to a function θ . Since $p \in (n-1, n)$, it follows from [14, Theorem 4.2], that \mathbf{u}^* satisfies condition INV, $\theta = \det \nabla \mathbf{u}^*$, and $\det \nabla \mathbf{u}^* > 0$ almost everywhere. Thus $\mathbf{u}^* \in \mathcal{A}_{\mathbf{A}}^\varepsilon$.

Upon adapting the lower continuity results in [18], it follows that $E_{\varepsilon, \mu, \eta}$ is sequentially weakly lower semicontinuous. Thus we have that

$$E_{\varepsilon, \mu, \eta}(\mathbf{u}^*) \leq \liminf_{k_j} E_{\varepsilon, \mu, \eta}(\mathbf{u}_{k_j}) = g^* = \inf_{\mathbf{u} \in \mathcal{A}_{\mathbf{A}}^\varepsilon} E_{\varepsilon, \mu, \eta}(\mathbf{u}),$$

i.e., that $\mathbf{u}_{V, \varepsilon, \mu, \eta} \equiv \mathbf{u}^* \in \mathcal{A}_{\mathbf{A}}^\varepsilon$ is a minimizer.

For the last part of the proposition, we argue by contradiction. Thus we assume that for some δ_0 there exists a sequence $\eta_j \rightarrow \infty$ such that the corresponding minimizers $\{\mathbf{u}_j\}$ satisfy $|c_\varepsilon(\mathbf{u}_j)| \geq \delta_0$ for all j . Note that for all j ,

$$E_{\varepsilon, \mu, \eta_j}(\mathbf{u}_j) \leq \begin{cases} \inf_{\mathbf{u} \in \mathcal{A}_{\mathbf{A}}^\varepsilon} & \int_{\Omega_\varepsilon} W(\nabla \mathbf{u}(\mathbf{x})) \, d\mathbf{x}, \\ \text{subject to} & c_\varepsilon(\mathbf{u}) = 0. \end{cases} \equiv f_\varepsilon^*, \quad (2.1)$$

Since $c_\varepsilon(\mathbf{u}_j) \geq -C$ with $C > 0$, if $\mu \geq 0$, we get that

$$f_\varepsilon^* \geq \mu c_\varepsilon(\mathbf{u}_j) + \frac{1}{2} \eta_j c_\varepsilon(\mathbf{u}_j)^2 \geq -\mu C + \frac{1}{2} \eta_j \delta_0^2 \rightarrow \infty,$$

as $j \rightarrow \infty$ which is a contradiction. If $\mu < 0$ and $\{c_\varepsilon(\mathbf{u}_j)\}$ is unbounded above, we can argue as before to get a contradiction as well. If $\{c_\varepsilon(\mathbf{u}_j)\}$ is bounded, then

$$f_\varepsilon^* \geq \mu c_\varepsilon(\mathbf{u}_j) + \frac{1}{2} \eta_j c_\varepsilon(\mathbf{u}_j)^2 \geq \mu c_\varepsilon(\mathbf{u}_j) + \frac{1}{2} \eta_j \delta_0^2 \rightarrow \infty,$$

which is a contradiction once again and this completes the proof. \square

We now show how to construct sequences $\{\mu_j\}$ and $\{\eta_j\}$ such that the computed minimizers in (1.12), converge to the solution of (1.8).

Theorem 2.2. *Let the stored energy function W satisfy the conditions H1–H2. Let $\gamma \in (0, 1)$, $\beta > 1$, $\eta_1 > 0$, $\mu_1 \in \mathbb{R}$, and $\mathbf{u}_0 \in \mathcal{A}_\mathbf{A}^\varepsilon$ be given. Let the sequences $\{\mu_j\}$, $\{\eta_j\}$, and $\{\mathbf{u}_j\}$ be given by:*

$$E_{\varepsilon, \mu_j, \eta_j}(\mathbf{u}_j) = \min_{\mathbf{u} \in \mathcal{A}_\mathbf{A}^\varepsilon} E_{\varepsilon, \mu_j, \eta_j}(\mathbf{u}), \quad (2.2a)$$

$$\mu_{j+1} = \mu_j + \eta_j c_\varepsilon(\mathbf{u}_j), \quad (2.2b)$$

$$\eta_{j+1} = \begin{cases} \eta_j & , \text{ if } |c_\varepsilon(\mathbf{u}_j)| \leq \gamma |c_\varepsilon(\mathbf{u}_{j-1})|, \\ \beta \eta_j & , \text{ otherwise.} \end{cases} \quad (2.2c)$$

Assume that $\{\mu_j\}$ is bounded. Then $c_\varepsilon(\mathbf{u}_j) \rightarrow 0$, and $\{\mathbf{u}_j\}$ has a subsequence $\{\mathbf{u}_{j_k}\}$ that converges weakly in $W^{1,p}(\Omega_\varepsilon)$ to $\mathbf{u}_\varepsilon \in \mathcal{A}_\mathbf{A}^\varepsilon$ where

$$E_\varepsilon(\mathbf{u}_\varepsilon) = \begin{cases} \min_{\mathbf{u} \in \mathcal{A}_\mathbf{A}^\varepsilon} \int_{\Omega_\varepsilon} W(\nabla \mathbf{u}(\mathbf{x})) \, d\mathbf{x}, \\ \text{subject to } c_\varepsilon(\mathbf{u}) = 0. \end{cases} \quad (2.3)$$

Proof: By Proposition 2.1, a function $\mathbf{u}_j \in \mathcal{A}_\mathbf{A}^\varepsilon$ satisfying (2.2a) exists for each j . From (2.1) we get that

$$E_{\varepsilon, \mu_j, \eta_j}(\mathbf{u}_j) \leq f_\varepsilon^*, \quad \forall j.$$

From this inequality and using that W is nonnegative, we get that

$$\mu_j c_\varepsilon(\mathbf{u}_j) + \frac{1}{2} \eta_j c_\varepsilon(\mathbf{u}_j)^2 \leq f_\varepsilon^*, \quad \forall j. \quad (2.4)$$

Note that the sequence $\{\eta_j\}$ is increasing. Thus in (2.2c) we have two possibilities:

- i) the sequence $\{\eta_j\}$ remains bounded, in which case, $|c_\varepsilon(\mathbf{u}_j)| \leq \gamma |c_\varepsilon(\mathbf{u}_{j-1})|$ is satisfied for all but finitely many indexes j . Clearly $c_\varepsilon(\mathbf{u}_j) \rightarrow 0$ in this case.
- ii) Otherwise (for a subsequence) $\eta_j \rightarrow \infty$, in which case (2.4) would imply that $c_\varepsilon(\mathbf{u}_j) \rightarrow 0$.

Thus in any case we have that $c_\varepsilon(\mathbf{u}_j) \rightarrow 0$.

If $\mu_j c_\varepsilon(\mathbf{u}_j) \geq -L$ for all j , where $L > 0$, then from (2.1) we get that

$$\int_{\Omega_\varepsilon} W(\nabla \mathbf{u}_j(\mathbf{x})) \, d\mathbf{x} - L \leq f_\varepsilon^*.$$

As in the proof of Proposition 2.1, there exists a subsequence $\{\mathbf{u}_{j_k}\}$ which converges weakly in $W^{1,p}(\Omega_\varepsilon)$ to a function \mathbf{u}_ε , that $\{\det \nabla \mathbf{u}_{j_k}\}$ converges weakly in $L^1(\Omega_\varepsilon)$ to $\det \nabla \mathbf{u}_\varepsilon$, \mathbf{u}_ε satisfies condition INV, and $\det \nabla \mathbf{u}_\varepsilon > 0$ almost everywhere. Thus $\mathbf{u}_\varepsilon \in \mathcal{A}_\mathbf{A}^\varepsilon$ and $c_\varepsilon(\mathbf{u}_\varepsilon) = 0$. Moreover, since $\mu_j c_\varepsilon(\mathbf{u}_j) \rightarrow 0$ by the assumed boundedness in $\{\mu_j\}$, we have that

$$f_\varepsilon^* \leq E_\varepsilon(\mathbf{u}_\varepsilon) \leq \liminf_k E_{\varepsilon, \mu_{j_k}, \eta_{j_k}}(\mathbf{u}_{j_k}) \leq f_\varepsilon^*.$$

It follows that

$$E_\varepsilon(\mathbf{u}_\varepsilon) = \begin{cases} \min_{\mathbf{u} \in \mathcal{A}_\mathbf{A}^\varepsilon} & \int_{\Omega_\varepsilon} W(\nabla \mathbf{u}(\mathbf{x})) \, d\mathbf{x}, \\ \text{subject to} & c_\varepsilon(\mathbf{u}) = 0. \end{cases}$$

□

Our next results give conditions under which the minimizer \mathbf{u}_ε in (2.3), satisfies a weak form of the Euler-Lagrange equations for this problem.

Theorem 2.3. *Let $\{\mathbf{u}_j\}$ be the sequence of Theorem 2.2 generated according to (2.2), and $\{\mathbf{u}_{j_k}\}$ a subsequence that converges weakly in $W^{1,p}(\Omega_\varepsilon)$ to a solution \mathbf{u}_ε of (2.3). Assume that there exist constants $K, \varepsilon_0 > 0$ such that the stored energy function W satisfies:*

$$\left| \frac{dW}{d\mathbf{F}}(\mathbf{C}\mathbf{F})\mathbf{F}^T \right| \leq K [W(\mathbf{F}) + 1] \quad \text{for all } \mathbf{F} \in \mathbf{M}_+^{n \times n}, \quad (2.5)$$

whenever $|\mathbf{C} - \mathbf{I}| < \varepsilon_0$. Then $\{\mu_j\}$ has a subsequence converging to μ_ε , where¹

$$\int_{\Omega_\varepsilon} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) + \mu_\varepsilon (\text{adj } \nabla \mathbf{u}_\varepsilon)^T \right] \cdot \nabla [\mathbf{v}(\mathbf{u}_\varepsilon)] \, d\mathbf{x} = 0, \quad (2.6)$$

for all $\mathbf{v} \in C^1(\mathbb{R}^n)$ with $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n \setminus \mathcal{E}$, where $\mathcal{E} = \{\mathbf{A}\mathbf{x} : \mathbf{x} \in \Omega\}$. Moreover if $\mathbf{u}_\varepsilon \in C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega}_\varepsilon)$ with $\det \nabla \mathbf{u}_\varepsilon > 0$ in Ω_ε , then

$$\text{div} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) + \mu_\varepsilon (\text{adj } \nabla \mathbf{u}_\varepsilon)^T \right] = \mathbf{0}, \quad \text{in } \Omega_\varepsilon, \quad (2.7a)$$

¹ μ_ε is the Lagrange multiplier corresponding to the volume constraint in (2.3) and is a measure of the Cauchy stress acting on the deformed inner cavity (cf. (3.11)).

$$\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad \text{on } \partial\Omega, \quad (2.7b)$$

$$\left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) + \mu_\varepsilon (\text{adj } \nabla \mathbf{u}_\varepsilon)^T \right] \cdot \mathbf{n} = \mathbf{0} \quad \text{on } \partial\mathcal{B}_\varepsilon(\mathbf{x}_0), \quad (2.7c)$$

$$\int_{\Omega_\varepsilon} \det \nabla \mathbf{u}_\varepsilon \, d\mathbf{x} = \frac{\omega_n}{n} \det \mathbf{A} - V. \quad (2.7d)$$

Proof: To show (2.6), we first derive the corresponding equilibrium equation for each \mathbf{u}_j . We use variations of \mathbf{u}_j of the form $\mathbf{u}_s = \mathbf{u}_j + s\mathbf{v}(\mathbf{u}_j)$ where $\mathbf{v} \in C^1(\mathbb{R}^n)$ with $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n \setminus \mathcal{E}$. From [18, Corollary 6.4] it follows that for s small enough, the function $\mathbf{u}_s \in \mathcal{A}_\mathbf{A}^\varepsilon$. (Note that the variation \mathbf{u}_s is not required to satisfy the constraint $c_\varepsilon(\mathbf{u}) = 0$ as \mathbf{u}_j is a solution of an unconstrained problem.) To show (2.6) for \mathbf{u}_j , first note that

$$\begin{aligned} & \int_{\Omega_\varepsilon} [W(\nabla \mathbf{u}_s) - W(\nabla \mathbf{u}_j)] \, d\mathbf{x} \\ &= \int_{\Omega_\varepsilon} \int_0^1 \frac{dW}{dt} (t\nabla \mathbf{u}_s + (1-t)\nabla \mathbf{u}_j) \, dt \, d\mathbf{x}, \\ &= \int_{\Omega_\varepsilon} \int_0^1 \frac{dW}{d\mathbf{F}} (t\nabla \mathbf{u}_s + (1-t)\nabla \mathbf{u}_j) \cdot (\nabla \mathbf{u}_s - \nabla \mathbf{u}_j) \, dt \, d\mathbf{x}, \\ &= s \int_{\Omega_\varepsilon} \left[\int_0^1 \frac{dW}{d\mathbf{F}} ([\mathbf{I} + st\nabla \mathbf{v}(\mathbf{u}_j)] \nabla \mathbf{u}_j) \nabla \mathbf{u}_j^T \, dt \right] \cdot \nabla \mathbf{v}(\mathbf{u}_j) \, d\mathbf{x} \end{aligned}$$

It follows now from (2.5) that for s small enough,

$$\left| \int_0^1 \frac{dW}{d\mathbf{F}} ([\mathbf{I} + st\nabla \mathbf{v}(\mathbf{u}_j)] \nabla \mathbf{u}_j) \nabla \mathbf{u}_j^T \, dt \right| \leq K[W(\nabla \mathbf{u}_j) + 1] \in L^1(\Omega_\varepsilon).$$

Upon invoking the Dominated Convergence Theorem, we get that

$$\lim_{s \rightarrow 0} \frac{1}{s} \int_{\Omega_\varepsilon} [W(\nabla \mathbf{u}_s) - W(\nabla \mathbf{u}_j)] \, d\mathbf{x} = \int_{\Omega_\varepsilon} \frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_j) \nabla \mathbf{u}_j^T \cdot \nabla \mathbf{v}(\mathbf{u}_j) \, d\mathbf{x} \quad (2.8)$$

Also

$$\begin{aligned} & \mu_j [c_\varepsilon(\mathbf{u}_s) - c_\varepsilon(\mathbf{u}_j)] + \frac{1}{2} \eta_j [c_\varepsilon^2(\mathbf{u}_s) - c_\varepsilon^2(\mathbf{u}_j)] \\ &= [\mu_j + \frac{1}{2} \eta_j (c_\varepsilon(\mathbf{u}_s) + c_\varepsilon(\mathbf{u}_j))] [c_\varepsilon(\mathbf{u}_s) - c_\varepsilon(\mathbf{u}_j)]. \end{aligned}$$

Now

$$c_\varepsilon(\mathbf{u}_s) - c_\varepsilon(\mathbf{u}_j) = \int_{\Omega_\varepsilon} (\det \nabla \mathbf{u}_s - \det \nabla \mathbf{u}_j) \, d\mathbf{x},$$

$$\begin{aligned}
&= \int_{\Omega_\varepsilon} \int_0^1 \frac{d}{dt} \det(t \nabla \mathbf{u}_s + (1-t) \nabla \mathbf{u}_j) dt d\mathbf{x}, \\
&= \int_{\Omega_\varepsilon} \int_0^1 [\text{adj}([\mathbf{I} + st \nabla \mathbf{v}(\mathbf{u}_j)] \nabla \mathbf{u}_j)]^T \cdot (\nabla \mathbf{u}_s - \nabla \mathbf{u}_j) dt d\mathbf{x}, \\
&= s \int_{\Omega_\varepsilon} \int_0^1 [\text{adj}(\mathbf{I} + st \nabla \mathbf{v}(\mathbf{u}_j))]^T (\text{adj} \nabla \mathbf{u}_j)^T \nabla \mathbf{u}_j^T \cdot \nabla \mathbf{v}(\mathbf{u}_j) dt d\mathbf{x}, \\
&= s \int_{\Omega_\varepsilon} \left[\int_0^1 [\text{adj}(\mathbf{I} + st \nabla \mathbf{v}(\mathbf{u}_j))]^T dt \right] \cdot \nabla \mathbf{v}(\mathbf{u}_j) \det \nabla \mathbf{u}_j d\mathbf{x}.
\end{aligned}$$

It follows now since $\mathbf{v} \in C^1(\mathbb{R}^n)$ with $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n \setminus \mathcal{E}$, that

$$\lim_{s \rightarrow 0} \frac{1}{s} [c_\varepsilon(\mathbf{u}_s) - c_\varepsilon(\mathbf{u}_j)] = \int_{\Omega_\varepsilon} [\mathbf{I} \cdot \nabla \mathbf{v}(\mathbf{u}_j)] \det \nabla \mathbf{u}_j d\mathbf{x}.$$

Combining this with (2.8) and using that $c_\varepsilon(\mathbf{u}_s) \rightarrow c_\varepsilon(\mathbf{u}_j)$ as $s \rightarrow 0$, we get that

$$\begin{aligned}
\left. \frac{d}{ds} E_{\varepsilon, \mu_j, \eta_j}(\mathbf{u}_s) \right|_{s=0} &= \int_{\Omega_\varepsilon} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_j) \nabla \mathbf{u}_j^T \right. \\
&\quad \left. + (\mu_j + \eta_j c_\varepsilon(\mathbf{u}_j)) (\det \nabla \mathbf{u}_j) \mathbf{I} \right] \cdot \nabla \mathbf{v}(\mathbf{u}_j) d\mathbf{x}.
\end{aligned}$$

Since \mathbf{u}_j is a minimizer, we must have that

$$\int_{\Omega_\varepsilon} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_j) \nabla \mathbf{u}_j^T + (\mu_j + \eta_j c_\varepsilon(\mathbf{u}_j)) (\det \nabla \mathbf{u}_j) \mathbf{I} \right] \cdot \nabla \mathbf{v}(\mathbf{u}_j) d\mathbf{x} = 0, \quad (2.9)$$

for all such \mathbf{v} 's. We now drop to the subsequence $\{\mathbf{u}_{j_k}\}$ that converges weakly in $W^{1,p}(\Omega_\varepsilon)$ to \mathbf{u}_ε and with $\det \nabla \mathbf{u}_{j_k} \rightharpoonup \det \nabla \mathbf{u}_\varepsilon$ in $L^1(\Omega_\varepsilon)$. Using (2.5) and the Dominated Convergence Theorem once again, we get that $\{\mu_{j_k}\}$ has a subsequence converging to μ_ε where

$$\int_{\Omega_\varepsilon} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) \nabla \mathbf{u}_\varepsilon^T + \mu_\varepsilon (\det \nabla \mathbf{u}_\varepsilon) \mathbf{I} \right] \cdot \nabla \mathbf{v}(\mathbf{u}_\varepsilon) d\mathbf{x} = 0.$$

Since $(\det \nabla \mathbf{u}_\varepsilon) \mathbf{I} = (\text{adj} \nabla \mathbf{u}_\varepsilon)^T \nabla \mathbf{u}_\varepsilon^T$ and $\nabla[\mathbf{v}(\mathbf{u}_\varepsilon)] = \nabla \mathbf{v}(\mathbf{u}_\varepsilon) \nabla \mathbf{u}_\varepsilon$, we get that the above equation can be written as

$$\int_{\Omega_\varepsilon} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) + \mu_\varepsilon (\text{adj} \nabla \mathbf{u}_\varepsilon)^T \right] \cdot \nabla[\mathbf{v}(\mathbf{u}_\varepsilon)] d\mathbf{x} = 0.$$

Now assume that $\mathbf{u}_\varepsilon \in C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega}_\varepsilon)$ with $\det \nabla \mathbf{u}_\varepsilon > 0$ in Ω_ε . Note that (2.7b) and (2.7d) follow from the fact that \mathbf{u}_ε is a solution of (2.3). The proof that (2.7a) holds is similar to the one given in [18, Theorem 5.1] and thus we omit it. Now multiply (2.7a) by $\mathbf{v}(\mathbf{u}_\varepsilon)$ where $\mathbf{v} \in C^1(\mathbb{R}^n)$ with $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n \setminus \mathcal{E}$, and integrate by parts using (2.6) to get that

$$\int_{\partial\Omega_\varepsilon} \left(\left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) + \mu_\varepsilon (\text{adj } \nabla \mathbf{u}_\varepsilon)^T \right] \cdot \mathbf{n} \right) \cdot \mathbf{v}(\mathbf{u}_\varepsilon) ds(\mathbf{x}) = 0.$$

Since the normal \mathbf{n} to $\partial\Omega_\varepsilon$ is mapped by \mathbf{u}_ε to

$$\tilde{\mathbf{n}}(\mathbf{u}_\varepsilon) = (\det \nabla \mathbf{u}_\varepsilon) (\nabla \mathbf{u}_\varepsilon)^{-T} \mathbf{n},$$

upon setting $\mathbf{y} = \mathbf{u}_\varepsilon(\mathbf{x})$, the previous equation is equivalent to:

$$\int_{\mathbf{u}_\varepsilon(\partial\Omega_\varepsilon)} ([\mathbf{T}(\mathbf{y}) + \mu_\varepsilon \mathbf{I}] \cdot \tilde{\mathbf{n}}(\mathbf{y})) \cdot \mathbf{v}(\mathbf{y}) ds(\mathbf{y}) = 0, \quad (2.10)$$

where the Cauchy stress tensor $\mathbf{T}(\mathbf{u}_\varepsilon)$ is given by

$$\mathbf{T}(\mathbf{u}_\varepsilon) = (\det \nabla \mathbf{u}_\varepsilon)^{-1} \frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) (\nabla \mathbf{u}_\varepsilon)^T.$$

Since $\mathbf{v} = \mathbf{0}$ on $\mathbf{u}_\varepsilon(\partial\Omega)$, we get that (2.10) implies that

$$[\mathbf{T}(\mathbf{y}) + \mu_\varepsilon \mathbf{I}] \cdot \tilde{\mathbf{n}}(\mathbf{y}) = \mathbf{0}, \quad \forall \mathbf{y} \in \mathbf{u}_\varepsilon(\partial\mathcal{B}_\varepsilon(\mathbf{x}_0)).$$

which after changing variables back to Ω_ε yields (2.7c). \square

We now study the sensitivity of the attained minimum value in (2.3) with respect to changes in the matrix \mathbf{A} . In the usual sensitivity theorems of optimization theory, the parameters that change are in the right hand sides of the constraints. In our problem however, the matrix \mathbf{A} appears both in the right hand side of the volume constraint and in the displacement boundary condition on $\partial\Omega$ (c.f. (2.7b), (2.7d)). Thus our calculation picks up an additional term from $\partial\Omega$. To emphasize the dependence of \mathbf{u}_ε on \mathbf{A} , we use the notation $\mathbf{u}_\varepsilon(\cdot, \mathbf{A})$.

Theorem 2.4. *Let $\mathbf{u}_\varepsilon(\cdot, \mathbf{A})$ be the minimizer in (2.3) and assume that $\mathbf{u}_\varepsilon(\cdot, \mathbf{A}) \in C^2(\Omega_\varepsilon) \cap C^1(\overline{\Omega}_\varepsilon)$ and that $\mathbf{u}_\varepsilon \in C^2(\Omega_\varepsilon \times M_+^{n \times n})$. Then for $\mathbf{A} = \text{diag}(\lambda_1, \dots, \lambda_n)$, $\lambda_i > 0$ for all i , we have that*

$$\frac{\partial}{\partial \lambda_i} E_\varepsilon(\mathbf{u}_\varepsilon(\cdot, \mathbf{A})) = \int_{\partial\Omega} x_i \mathbf{e}_i \cdot \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) + \mu_\varepsilon (\text{adj } \nabla \mathbf{u}_\varepsilon)^T \right] \cdot \mathbf{n} ds$$

$$-\mu_\varepsilon \frac{\omega_n \det \mathbf{A}}{n\lambda_i}, \quad i = 1, \dots, n, \quad (2.11)$$

where $\{\mathbf{e}_k\}$ is the standard basis of \mathbb{R}^n .

Proof: Let $\mathbf{u}_{\varepsilon,i} = \frac{\partial \mathbf{u}_\varepsilon}{\partial \lambda_i}$. By the assumed smoothness on \mathbf{u}_ε , we have that

$$\frac{\partial}{\partial \lambda_i} E_\varepsilon(\mathbf{u}_\varepsilon(\cdot, \mathbf{A})) = \frac{\partial}{\partial \lambda_i} \int_{\Omega_\varepsilon} W(\nabla \mathbf{u}_\varepsilon) \, d\mathbf{x} = \int_{\Omega_\varepsilon} \frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) \cdot \nabla \mathbf{u}_{\varepsilon,i} \, d\mathbf{x}. \quad (2.12)$$

If we multiply (2.7a) by $\mathbf{u}_{\varepsilon,i}$, integrate by parts, and use the boundary condition (2.7c), we get that

$$\begin{aligned} \int_{\Omega_\varepsilon} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) + \mu_\varepsilon (\text{adj } \nabla \mathbf{u}_\varepsilon)^T \right] \cdot \nabla \mathbf{u}_{\varepsilon,i} \, d\mathbf{x} \\ = \int_{\partial\Omega} \left(\left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) + \mu_\varepsilon (\text{adj } \nabla \mathbf{u}_\varepsilon)^T \right] \cdot \mathbf{n} \right) \cdot \mathbf{u}_{\varepsilon,i} \, ds. \end{aligned}$$

Since (2.7b) implies that $\mathbf{u}_{\varepsilon,i}(\mathbf{x}) = x_i \mathbf{e}_i$ for $\mathbf{x} \in \partial\Omega$, the above equation can be written as

$$\begin{aligned} \int_{\Omega_\varepsilon} \frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) \cdot \nabla \mathbf{u}_{\varepsilon,i} \, d\mathbf{x} &= \int_{\partial\Omega} x_i \left(\left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_\varepsilon) + \mu_\varepsilon (\text{adj } \nabla \mathbf{u}_\varepsilon)^T \right] \cdot \mathbf{n} \right) \cdot \mathbf{e}_i \, ds \\ &\quad - \mu_\varepsilon \int_{\Omega_\varepsilon} (\text{adj } \nabla \mathbf{u}_\varepsilon)^T \cdot \nabla \mathbf{u}_{\varepsilon,i} \, d\mathbf{x}. \end{aligned} \quad (2.13)$$

We now differentiate (2.7d) with respect to λ_i to get that

$$\int_{\Omega_\varepsilon} (\text{adj } \nabla \mathbf{u}_\varepsilon)^T \cdot \nabla \mathbf{u}_{\varepsilon,i} \, d\mathbf{x} = \frac{\omega_n}{n} \frac{\partial}{\partial \lambda_i} (\det \mathbf{A}) = \frac{\omega_n \det \mathbf{A}}{n\lambda_i}.$$

Combining this with (2.12) and (2.13), gives the result (2.11). \square

3 Convergence of the regularized constrained minimizers

We now show that the regularized constrained minimizers whose existence is given by Theorem 2.2, converge to a solution of the “non-regular” constrained problem (1.6). The first part of the proof of this result, dealing with the convergence and the existence of

the limit, is very similar to that in [19, Theorem 4.1] and consequently we sketch most of it. The second part in which we show that the limiting function is actually a solution of (1.6) is more subtle, again due to the treatment of the integral volume constraint in (1.6). In particular we need the following result which shows that given any function $\mathbf{u} \in \mathcal{C}_{\mathbf{A}}^0$ and any sequence $\varepsilon_j \rightarrow 0$, we can approximate \mathbf{u} with a sequence of functions $\{\hat{\mathbf{u}}_j\}$ where $\hat{\mathbf{u}}_j \in \mathcal{C}_{\mathbf{A}}^{\varepsilon_j}$ for all j .

Lemma 3.1. *Let Ω be a bounded, open, convex set, and let the stored energy function W satisfy the conditions H1–H2, and assume further that for some $K, \gamma_0 > 0$,*

$$|W(\mathbf{F}\mathbf{C})| \leq K[W(\mathbf{F}) + 1], \quad (3.1)$$

whenever $\|\mathbf{C} - \mathbf{I}\| < \gamma_0$. Let $\mathbf{u} \in \mathcal{C}_{\mathbf{A}}^0$. Then for any sequence $\{\varepsilon_j\}$ with $\varepsilon_j \rightarrow 0$, there exists a sequence of functions $\{\hat{\mathbf{u}}_j\}$ in $W^{1,p}(\Omega)$ such that

$$\lim_{j \rightarrow \infty} \int_{\Omega} W(\nabla \hat{\mathbf{u}}_j(\mathbf{x})) \, d\mathbf{x} = \int_{\Omega} W(\nabla \mathbf{u}(\mathbf{x})) \, d\mathbf{x}.$$

Moreover, $\hat{\mathbf{u}}_j|_{\Omega_{\varepsilon_j}} \in \mathcal{C}_{\mathbf{A}}^{\varepsilon_j}$.

Proof: For any $0 < \eta < 1$ and $\varepsilon > 0$, we let

$$\Omega_{\varepsilon}^{\eta} = \{\mathbf{x} \in \Omega_{\varepsilon} : \text{dist}(\mathbf{x}, \partial\Omega) > 1 - \eta\}.$$

In terms of our previous notation, we have that $\Omega_{\varepsilon} = \Omega_{\varepsilon}^1$. Now $\partial\Omega_{\varepsilon}^{\eta} = \partial\mathcal{B}_{\varepsilon}(\mathbf{x}_0) \cup \omega_{\eta}$ where

$$\omega_{\eta} = \{\mathbf{x} \in \Omega_{\varepsilon} : \text{dist}(\mathbf{x}, \partial\Omega) = 1 - \eta\}.$$

For each $\mathbf{y} \in \omega_{\eta}$ there exists a unique $\mathbf{x}(\mathbf{y}) \in \partial\mathcal{B}_{\varepsilon}(\mathbf{x}_0)$ such that

$$\|\mathbf{x}(\mathbf{y}) - \mathbf{y}\| = \text{dist}(\mathbf{y}, \partial\mathcal{B}_{\varepsilon}(\mathbf{x}_0)).$$

Since Ω is convex, the segment

$$]\mathbf{x}(\mathbf{y}), \mathbf{y}[= \{\gamma\mathbf{x}(\mathbf{y}) + (1 - \gamma)\mathbf{y} : 0 < \gamma < 1\},$$

belongs to $\Omega_{\varepsilon}^{\eta}$. Moreover, there exists a unique $\mathbf{z}(\mathbf{y}) \in \partial\Omega$ such that

$$]\mathbf{x}(\mathbf{y}), \mathbf{y}[\subset]\mathbf{x}(\mathbf{y}), \mathbf{z}(\mathbf{y})[\subset \Omega_{\varepsilon}.$$

Since the segments $]\mathbf{x}(\mathbf{y}), \mathbf{y}[$, $]\mathbf{x}(\mathbf{y}), \mathbf{z}(\mathbf{y})[$ can be put into a one to one correspondence, after letting $\mathbf{y} \in \omega_\eta$ to vary over ω_η , this basically shows that we can construct a diffeomorphism $\mathbf{\Gamma} : \Omega_\varepsilon \rightarrow \Omega_\varepsilon^\eta$ such that $\det \nabla \mathbf{\Gamma} > 0$ over Ω_ε and with $\mathbf{\Gamma}(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x} \in \partial \mathcal{B}_\varepsilon(\mathbf{x}_0)$. For any $\mathbf{u} \in \mathcal{C}_\mathbf{A}^0$, define $\hat{\mathbf{u}}_\eta$ on Ω_ε by:

$$\hat{\mathbf{u}}_\eta(\mathbf{y}) = \begin{cases} \mathbf{u}(\mathbf{\Gamma}^{-1}(\mathbf{y})), & \mathbf{y} \in \Omega_\varepsilon^\eta, \\ \mathbf{A}\mathbf{y}, & \mathbf{y} \in \Omega_\varepsilon \setminus \Omega_\varepsilon^\eta. \end{cases}$$

We now show that $\hat{\mathbf{u}}_\eta \in \mathcal{C}_\mathbf{A}^\varepsilon$ for η (depending on ε) sufficiently close to 1.

i) Clearly $\hat{\mathbf{u}}_\eta(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for $\mathbf{x} \in \partial \Omega$, and

$$\det [\nabla(\mathbf{u}(\mathbf{\Gamma}^{-1}(\mathbf{y})))] = \det \nabla \mathbf{u}(\mathbf{\Gamma}^{-1}(\mathbf{y})) \det \nabla \mathbf{\Gamma}^{-1}(\mathbf{y}) > 0,$$

a.e. on Ω_ε since $\det \nabla \mathbf{u} > 0$ a.e. on Ω , and $\det \nabla \mathbf{\Gamma} > 0$ implies that $\det \nabla \mathbf{\Gamma}^{-1} > 0$ on Ω_ε . Clearly $\hat{\mathbf{u}}_\eta$ satisfies INV in Ω_ε and since $\delta_{\mathbf{x}_0}(\Omega_\varepsilon) = 0$, it follows that $\hat{\mathbf{u}}_\eta \in \mathcal{A}_\mathbf{A}^\varepsilon$.

ii) Since $\mathbf{u} \in \mathcal{C}_\mathbf{A}^0$, we have that

$$\int_{\Omega} \det \nabla \mathbf{u} \, d\mathbf{x} = (\det \mathbf{A}) |\Omega| - V.$$

Hence

$$\int_{\Omega_\varepsilon} \det \nabla \mathbf{u} \, d\mathbf{x} = (\det \mathbf{A}) |\Omega| - V_\varepsilon, \quad V_\varepsilon = V + \int_{\mathcal{B}_\varepsilon(\mathbf{x}_0)} \det \nabla \mathbf{u} \, d\mathbf{x}.$$

Note that $V_\varepsilon > V$, and $V_\varepsilon \searrow V$ as $\varepsilon \searrow 0$. Now

$$\begin{aligned} \int_{\Omega_\varepsilon} \det \nabla \hat{\mathbf{u}}_\eta \, d\mathbf{y} &= \int_{\Omega_\varepsilon^\eta} \det \nabla \mathbf{u}(\mathbf{\Gamma}^{-1}(\mathbf{y})) \det \nabla \mathbf{\Gamma}^{-1}(\mathbf{y}) \, d\mathbf{y} \\ &\quad + \int_{\Omega_\varepsilon \setminus \Omega_\varepsilon^\eta} \det \mathbf{A} \, d\mathbf{y}, \\ &= \int_{\Omega_\varepsilon} \det \nabla \mathbf{u} \, d\mathbf{x} + (\det \mathbf{A}) (|\Omega| - |\Omega^\eta|), \\ &= (\det \mathbf{A}) |\Omega| - V_\varepsilon + (\det \mathbf{A}) (|\Omega| - |\Omega^\eta|), \end{aligned}$$

where

$$\Omega^\eta = \{\mathbf{x} \in \Omega : \text{dist}(\mathbf{x}, \partial \Omega) > 1 - \eta\}.$$

With ε fixed, choose $\eta(\varepsilon)$ such that

$$(\det \mathbf{A}) (|\Omega| - |\Omega^{\eta(\varepsilon)}|) = V_\varepsilon - V.$$

Note that $\eta(\varepsilon) \nearrow 1$ as $\varepsilon \searrow 0$. With this choice of $\eta(\varepsilon)$, we get that

$$\int_{\Omega_\varepsilon} \det \nabla \hat{\mathbf{u}}_{\eta(\varepsilon)} \, d\mathbf{y} = (\det \mathbf{A}) |\Omega| - V.$$

Combining this with result of part (i), we get that $\hat{\mathbf{u}}_{\eta(\varepsilon)} \in \mathcal{C}_{\mathbf{A}}^\varepsilon$.

Henceforth we set $\hat{\mathbf{u}}_\varepsilon = \hat{\mathbf{u}}_{\eta(\varepsilon)}$. Since $\mathbf{\Gamma}(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x} \in \partial\mathcal{B}_\varepsilon(\mathbf{x}_0)$, we can extend $\hat{\mathbf{u}}_\varepsilon$ to Ω by setting $\hat{\mathbf{u}}_\varepsilon(\mathbf{x}) = \mathbf{u}(\mathbf{x})$ for $\mathbf{x} \in B_\varepsilon(\mathbf{x}_0)$ with the resulting $\hat{\mathbf{u}}_\varepsilon$ now in $W^{1,p}(\Omega)$ and with $\hat{\mathbf{u}}_\varepsilon|_{\Omega_\varepsilon} \in \mathcal{C}_{\mathbf{A}}^\varepsilon$.

Now take $\{\varepsilon_j\}$ with $\varepsilon_j \rightarrow 0$ and set $\hat{\mathbf{u}}_j = \hat{\mathbf{u}}_{\varepsilon_j}$. Let $\mathbf{\Gamma}_j : \Omega_{\varepsilon_j} \rightarrow \Omega_{\varepsilon_j}^{\eta(\varepsilon_j)}$ be the corresponding diffeomorphism in the definition of $\hat{\mathbf{u}}_j$. Since $\mathbf{\Gamma}_j$ when restricted to $\partial\mathcal{B}_{\varepsilon_j}(\mathbf{x}_0)$ is equal to the identity mapping, it follows that it can be extended continuously into $\mathcal{B}_{\varepsilon_j}$ as the identity. Thus $\nabla \mathbf{\Gamma}_j^{-1} \rightarrow \mathbf{I}$ as $j \rightarrow \infty$ in $L^\infty(\Omega)$. These observations together with (3.1) imply that

$$W(\nabla \hat{\mathbf{u}}_j) \leq K[W(\nabla \mathbf{u}) + 1],$$

a.e. in Ω for j large enough. Now $\nabla \hat{\mathbf{u}}_j \rightarrow \nabla \mathbf{u}$ a.e. in Ω . The above inequality and the Dominated Convergence Theorem can be used now to conclude that

$$\lim_{j \rightarrow \infty} \int_{\Omega} W(\nabla \hat{\mathbf{u}}_j(\mathbf{x})) \, d\mathbf{x} = \int_{\Omega} W(\nabla \mathbf{u}(\mathbf{x})) \, d\mathbf{x}.$$

□

We now have the main result of this paper.

Theorem 3.2. *Let the hypotheses in Lemma 3.1 hold. For $V \in (0, (\omega_n/n) \det \mathbf{A})$, let $\{\varepsilon_j\}$ be a sequence of positive numbers converging to zero, and for each ε_j , let \mathbf{u}_j be the corresponding minimizer given by Theorem 2.2 and satisfying (2.3). Then $\{\mathbf{u}_j\}$ has a subsequence $\{\mathbf{u}_{j_k}\}$ such that for any $\delta > 0$,*

$$\mathbf{u}_{j_k} \rightharpoonup \mathbf{u}_V \quad \text{in } W^{1,p}(\Omega_\delta),$$

where the function \mathbf{u}_V is a solution of (1.6), and with

$$E(\mathbf{u}_V) = \lim_k E_{\varepsilon_{j_k}}(\mathbf{u}_{j_k}).$$

Proof: We let

$$\mathcal{C}_{\mathbf{A}}^\varepsilon \equiv \{\mathbf{u} \in \mathcal{A}_{\mathbf{A}}^\varepsilon \mid c_\varepsilon(\mathbf{u}) = 0\}, \quad \varepsilon \geq 0.$$

It follows from Lemma B.1 that these sets are non empty for ε small enough. Thus each \mathbf{u}_j satisfies:

$$E_{\varepsilon_j}(\mathbf{u}_j) = \min_{\mathbf{u} \in \mathcal{C}_{\mathbf{A}}^{\varepsilon_j}} \int_{\Omega_{\varepsilon_j}} W(\nabla \mathbf{u}(\mathbf{x})) \, d\mathbf{x} = \min_{\mathbf{u} \in \mathcal{C}_{\mathbf{A}}^{\varepsilon_j}} E_{\varepsilon_j}(\mathbf{u}).$$

Now we fix an index $J \in \mathbb{N}$ and take $j > J$. It follows from hypothesis (H2) on W that for some constants $c_1 > 0$ and $c_2 \in \mathbb{R}$:

$$E_{\varepsilon_j}(\mathbf{u}_j) \geq c_1 \|\nabla \mathbf{u}_j\|_{L^p(\Omega_{\varepsilon_j})}^p + c_2, \quad j > J.$$

Again, it follows from (H2) that we may assume that W is non negative. Hence

$$E_{\varepsilon_j}(\mathbf{u}_j) \leq E_{\varepsilon_j}(\mathbf{u}_j) \leq C, \quad j > J,$$

where the constant C is given by Lemma B.1. Combining this with the inequality above, and since $\mathbf{u}_j = \mathbf{A}\mathbf{x}$ on $\partial\Omega$, we get that (for a subsequence) $\{\mathbf{u}_j\}$ converges weakly in $W^{1,p}(\Omega_{\varepsilon_j})$ to a function \mathbf{u}^J , and that $\{\det \nabla \mathbf{u}_j\}$ converges weakly in $L^1(\Omega_{\varepsilon_j})$ to a function θ^J . Since $p \in (n-1, n)$, it follows from [14, Theorem 4.2], that \mathbf{u}^J satisfies condition INV, $\theta^J = \det \nabla \mathbf{u}^J$, and $\det \nabla \mathbf{u}^J > 0$ almost everywhere. By choosing an appropriate diagonal sequence, it is shown in [19] that there exists a subsequence $\{\mathbf{u}_{j_k}\}$ and a function $\mathbf{u}_V \in W^{1,p}(\Omega)$ such that

$$\mathbf{u}_{j_k} \rightharpoonup \mathbf{u}_V, \quad \text{in } W^{1,p}(\Omega_{\varepsilon_j}).$$

The results in [19, Section 4.2] show that $\mathbf{u}_V \in \mathcal{A}_{\mathbf{A}}$.

It remains to show that \mathbf{u}_V is a solution of (1.6). By the results quoted in the previous paragraph, we get that the subsequence $\{\mathbf{u}_{j_k}\}$ has the property that

$$\det \nabla \mathbf{u}_{j_k} \rightharpoonup \det \nabla \mathbf{u}_V, \quad \text{in } L^1(\Omega_{\varepsilon_j}).$$

Since $\mathbf{u}_{j_k} \in \mathcal{C}_{\mathbf{A}}^{\varepsilon_{j_k}}$, we also have that

$$\int_{\Omega_{\varepsilon_{j_k}}} \det \nabla \mathbf{u}_{j_k} \, d\mathbf{x} = (\det \mathbf{A}) |\Omega| - V, \quad \forall k.$$

Now we extend $\det \nabla \mathbf{u}_{j_k}$ to Ω as follows:

$$g_k(\mathbf{x}) = \begin{cases} \det \nabla \mathbf{u}_{j_k}(\mathbf{x}), & \mathbf{x} \in \Omega_{\varepsilon_{j_k}}, \\ 0, & \mathbf{x} \in \Omega \setminus \Omega_{\varepsilon_{j_k}}. \end{cases}$$

Clearly $g_k \in L^1(\Omega)$ and since $\det \nabla \mathbf{u}_{j_k} > 0$ a.e. in $\Omega_{\varepsilon_{j_k}}$, we get that

$$\|g_k\|_{L^1(\Omega)} = \int_{\Omega} g_k \, d\mathbf{x} = \int_{\Omega_{\varepsilon_{j_k}}} \det \nabla \mathbf{u}_{j_k} \, d\mathbf{x} = (\det \mathbf{A}) |\Omega| - V, \quad \forall k.$$

Writing

$$\int_{\Omega} (\det \nabla \mathbf{u}_V - g_k) \, d\mathbf{x} = \int_{\Omega_{\varepsilon_J}} (\det \nabla \mathbf{u}_V - g_k) \, d\mathbf{x} + \int_{\Omega \setminus \Omega_{\varepsilon_J}} (\det \nabla \mathbf{u}_V - g_k) \, d\mathbf{x},$$

we observe that the second term on the right is bounded by a constant times $|\Omega \setminus \Omega_{\varepsilon_J}|$, and thus can be made arbitrarily small by taking J sufficiently large. Once J is fixed, the first term can be made arbitrarily small for k sufficiently large, as $g_k = \det \nabla \mathbf{u}_{j_k}$ over Ω_J for k sufficiently large and by the weak convergence of $\{\det \nabla \mathbf{u}_{j_k}\}$ to $\det \nabla \mathbf{u}_V$ in $L^1(\Omega_J)$. This essentially shows that

$$\int_{\Omega} \det \nabla \mathbf{u}_V \, d\mathbf{x} = \lim_{k \rightarrow \infty} \int_{\Omega} g_k \, d\mathbf{x} = (\det \mathbf{A}) |\Omega| - V,$$

Hence $\mathbf{u}_V \in \mathcal{C}_{\mathbf{A}}^0$. We now show that \mathbf{u}_V is a minimizer over $\mathcal{C}_{\mathbf{A}}^0$.

For any $\mathbf{u} \in \mathcal{C}_{\mathbf{A}}^0$ and for the subsequence $\{\varepsilon_{j_k}\}$ above, let $\{\hat{\mathbf{u}}_{j_k}\}$ be the corresponding sequence given by Lemma 3.1 with the property that

$$\lim_{k \rightarrow \infty} \int_{\Omega} W(\nabla \hat{\mathbf{u}}_{j_k})(\mathbf{x}) \, d\mathbf{x} = \int_{\Omega} W(\nabla \mathbf{u})(\mathbf{x}) \, d\mathbf{x}. \quad (3.2)$$

As a function over $\Omega_{\varepsilon_{j_k}}$, we have that $\hat{\mathbf{u}}_{j_k} \in \mathcal{C}_{\mathbf{A}}^{\varepsilon_{j_k}}$. Since \mathbf{u}_{j_k} is the minimizer over $\mathcal{C}_{\mathbf{A}}^{\varepsilon_{j_k}}$, we have that

$$\int_{\Omega_{\varepsilon_{j_k}}} W(\nabla \mathbf{u}_{j_k})(\mathbf{x}) \, d\mathbf{x} \leq \int_{\Omega_{\varepsilon_{j_k}}} W(\nabla \hat{\mathbf{u}}_{j_k})(\mathbf{x}) \, d\mathbf{x}. \quad (3.3)$$

Let $N > 0$ be given. For $k > N$ and the nonnegativity of W we get that

$$\int_{\Omega_{\varepsilon_{j_N}}} W(\nabla \mathbf{u}_{j_k})(\mathbf{x}) \, d\mathbf{x} \leq \int_{\Omega_{\varepsilon_{j_k}}} W(\nabla \mathbf{u}_{j_k})(\mathbf{x}) \, d\mathbf{x}. \quad (3.4)$$

By the results in [2], the functional $E_{\varepsilon_{j_N}}(\cdot)$ (cf. (1.9)) is weakly lower semi-continuous over $\mathcal{A}_{\mathbf{A}}^{\varepsilon_{j_N}}$. Using this and since $\mathbf{u}_{j_k} \rightharpoonup \mathbf{u}_V$ in $W^{1,p}(\Omega_{\varepsilon_{j_N}})$, we conclude that

$$\int_{\Omega_{\varepsilon_{j_N}}} W(\nabla \mathbf{u}_V)(\mathbf{x}) \, d\mathbf{x} \leq \liminf_k \int_{\Omega_{\varepsilon_{j_N}}} W(\nabla \mathbf{u}_{j_k})(\mathbf{x}) \, d\mathbf{x}. \quad (3.5)$$

From the nonnegativity of W , it follows from (3.2)–(3.5) that

$$\int_{\Omega_{\varepsilon_{j_N}}} W(\nabla \mathbf{u}_V(\mathbf{x})) \, d\mathbf{x} \leq \int_{\Omega} W(\nabla \mathbf{u}(\mathbf{x})) \, d\mathbf{x}.$$

Since N is arbitrary, we can conclude that

$$\int_{\Omega} W(\nabla \mathbf{u}_V(\mathbf{x})) \, d\mathbf{x} \leq \int_{\Omega} W(\nabla \mathbf{u}(\mathbf{x})) \, d\mathbf{x}.$$

Since $\mathbf{u} \in \mathcal{C}_{\mathbf{A}}^0$ is arbitrary, we get that \mathbf{u}_V is a minimizer over $\mathcal{C}_{\mathbf{A}}^0$. If we set $\mathbf{u} = \mathbf{u}_V$ in (3.2), then we get as well that

$$\int_{\Omega} W(\nabla \mathbf{u}_V(\mathbf{x})) \, d\mathbf{x} = \liminf_k \int_{\Omega_{\varepsilon_{j_k}}} W(\nabla \mathbf{u}_{j_k}(\mathbf{x})) \, d\mathbf{x},$$

from which the result about the energies follows upon taking another subsequence. \square

We now derive an expression for a weak form of the equilibrium equations for the minimizer \mathbf{u}_V in Theorem 3.2.

Theorem 3.3. *Assume that (2.5) and the hypotheses in Theorem 3.2 hold. Let \mathbf{u}_V be the minimizer in Theorem 3.2. Then there exists $\mu_V \in \mathbb{R}$ such that*

$$\int_{\Omega} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_V) + \mu_V (\text{adj } \nabla \mathbf{u}_V)^T \right] \cdot \nabla[\mathbf{v}(\mathbf{u}_V)] \, d\mathbf{x} = 0, \quad (3.6)$$

for all $\mathbf{v} \in C^1(\mathbb{R}^n)$ with $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n \setminus \mathcal{E}$, where $\mathcal{E} = \{\mathbf{Ax} : \mathbf{x} \in \Omega\}$. Moreover, if $\mathbf{u}_V \in C^2(\Omega \setminus \{\mathbf{x}_0\}) \cap C^1(\overline{\Omega} \setminus \{\mathbf{x}_0\})$ with $\det \nabla \mathbf{u}_V > 0$ in $\Omega \setminus \{\mathbf{x}_0\}$, then

$$\text{div} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_V) + \mu_V (\text{adj } \nabla \mathbf{u}_V)^T \right] = \mathbf{0}, \quad \text{in } \Omega \setminus \{\mathbf{x}_0\}, \quad (3.7)$$

and

$$\lim_{\delta \rightarrow 0} \int_{\partial \mathcal{B}_{\delta}(\mathbf{x}_0)} \left(\left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_V) + \mu_V (\text{adj } \nabla \mathbf{u}_V)^T \right] \cdot \mathbf{n} \right) \cdot \mathbf{v}(\mathbf{u}_V) \, ds(\mathbf{x}) = 0. \quad (3.8)$$

Proof: Let $\{\mathbf{u}_{j_k}\}$ be the subsequence given by Theorem 3.2 such that for any $\delta > 0$,

$$\mathbf{u}_{j_k} \rightharpoonup \mathbf{u}_V \quad \text{in } W^{1,p}(\Omega_{\delta}), \quad (3.9a)$$

$$\det \nabla \mathbf{u}_{j_k} \rightharpoonup \det \nabla \mathbf{u}_V \quad \text{in } L^1(\Omega_{\delta}), \quad (3.9b)$$

and with

$$E(\mathbf{u}_V) = \lim_k E_{\varepsilon_{j_k}}(\mathbf{u}_{j_k}). \quad (3.10)$$

(We recall that \mathbf{u}_{j_k} is the corresponding minimizer given by Theorem 2.2 corresponding to ε_{j_k} .) From Theorem 2.3 we have that

$$\int_{\Omega_{\varepsilon_{j_k}}} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_{\varepsilon_{j_k}}) \nabla \mathbf{u}_{\varepsilon_{j_k}}^T + \mu_{\varepsilon_{j_k}} (\det \nabla \mathbf{u}_{\varepsilon_{j_k}}) \mathbf{I} \right] \cdot \nabla \mathbf{v}(\mathbf{u}_{\varepsilon_{j_k}}) \, d\mathbf{x} = 0.$$

Let $\mathbf{g}_{\varepsilon_{j_k}}$ and $\mathbf{h}_{\varepsilon_{j_k}}$ represent the extensions (by the zero tensor) over Ω of the functions

$$\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_{\varepsilon_{j_k}}) \nabla \mathbf{u}_{\varepsilon_{j_k}}^T, \quad (\det \nabla \mathbf{u}_{\varepsilon_{j_k}}) \mathbf{I},$$

respectively. The sequence $\{\mathbf{u}_{\varepsilon_{j_k}}\}$ can be extended (cf. [19, p. 746]) as well to a sequence $\{\tilde{\mathbf{u}}_{\varepsilon_{j_k}}\}$ over Ω in such way that (3.9) holds now over Ω , with $\tilde{\mathbf{u}}_{\varepsilon_{j_k}} = \mathbf{u}_{\varepsilon_{j_k}}$ over $\Omega_{\varepsilon_{j_k}}$, and with $\tilde{\mathbf{u}}_{\varepsilon_{j_k}}(\Omega) \subset \mathcal{E}$. Using (2.5), (3.9). and (3.10) together with the Dominated convergence Theorem, we get that

$$\begin{aligned} \int_{\Omega} \frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_V) \nabla \mathbf{u}_V^T \cdot \nabla \mathbf{v}(\mathbf{u}_V) \, d\mathbf{x} &= \lim_{k \rightarrow \infty} \int_{\Omega} \mathbf{g}_{\varepsilon_{j_k}} \cdot \nabla \mathbf{v}(\tilde{\mathbf{u}}_{\varepsilon_{j_k}}) \, d\mathbf{x}, \\ \int_{\Omega} (\det \nabla \mathbf{u}_V) \mathbf{I} \cdot \nabla \mathbf{v}(\mathbf{u}_V) \, d\mathbf{x} &= \lim_{k \rightarrow \infty} \int_{\Omega} \mathbf{h}_{\varepsilon_{j_k}} \cdot \nabla \mathbf{v}(\tilde{\mathbf{u}}_{\varepsilon_{j_k}}) \, d\mathbf{x}. \end{aligned}$$

But

$$\begin{aligned} 0 &= \int_{\Omega_{\varepsilon_{j_k}}} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_{\varepsilon_{j_k}}) \nabla \mathbf{u}_{\varepsilon_{j_k}}^T + \mu_{\varepsilon_{j_k}} (\det \nabla \mathbf{u}_{\varepsilon_{j_k}}) \mathbf{I} \right] \cdot \nabla \mathbf{v}(\mathbf{u}_{\varepsilon_{j_k}}) \, d\mathbf{x} \\ &= \int_{\Omega} \left[\mathbf{g}_{\varepsilon_{j_k}} + \mu_{\varepsilon_{j_k}} \mathbf{h}_{\varepsilon_{j_k}} \right] \cdot \nabla \mathbf{v}(\tilde{\mathbf{u}}_{\varepsilon_{j_k}}) \, d\mathbf{x}, \end{aligned}$$

for all k . Combining this with the two limits above we that there exists $\mu_V \in \mathbb{R}$ such that $\mu_{\varepsilon_{j_k}} \rightarrow \mu_V$ and with

$$\int_{\Omega} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_V) \nabla \mathbf{u}_V^T + \mu_V (\det \nabla \mathbf{u}_V) \mathbf{I} \right] \cdot \nabla \mathbf{v}(\mathbf{u}_V) \, d\mathbf{x} = 0,$$

from which (3.6) follows.

Now assume that $\mathbf{u}_V \in C^2(\Omega \setminus \{\mathbf{x}_0\}) \cap C^1(\overline{\Omega} \setminus \{\mathbf{x}_0\})$ with $\det \nabla \mathbf{u}_V > 0$ in $\Omega \setminus \{\mathbf{x}_0\}$. The proof of (3.7) is similar to the one given in [18, Theorem 5.1] and thus we omit it. Let $\delta > 0$ be given. If we multiply (3.7) by $\mathbf{v}(\mathbf{u}_V)$, where $\mathbf{v} \in C^1(\mathbb{R}^n)$ with $\mathbf{v} = \mathbf{0}$ on $\mathbb{R}^n \setminus \mathcal{E}$, and integrate by parts over Ω_δ , we get that

$$\begin{aligned} & \int_{\Omega_\delta} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_V) + \mu_V (\text{adj } \nabla \mathbf{u}_V)^T \right] \cdot \nabla [\mathbf{v}(\mathbf{u}_V)] \, d\mathbf{x} \\ &= \int_{\partial \mathcal{B}_\delta(\mathbf{x}_0)} \left(\left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_V) + \mu_V (\text{adj } \nabla \mathbf{u}_V)^T \right] \cdot \mathbf{n} \right) \cdot \mathbf{v}(\mathbf{u}_V) \, ds(\mathbf{x}). \end{aligned}$$

Taking the limit as $\delta \searrow 0$ and using (3.6) we get that (3.8) holds. \square

Remark 3.4. One could try to prove Theorem 3.3 by using variations directly onto the functional in (1.6). However this approach would require constructing variations that preserve the volume constraint $c_0(\mathbf{u}) = 0$ (cf. (1.7)). Our proof using Theorem 2.3 avoids the technical complications of constructing such variations.

Remark 3.5. In terms of the Cauchy stress tensor:

$$\mathbf{T}(\mathbf{u}_V) = (\det \nabla \mathbf{u}_V)^{-1} \frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_V)(\nabla \mathbf{u}_V)^T,$$

(3.8) is equivalent to:

$$\lim_{\delta \rightarrow 0} \int_{\mathbf{u}_V(\partial \mathcal{B}_\delta(\mathbf{x}_0))} ([\mathbf{T}(\mathbf{y}) + \mu_V \mathbf{I}] \cdot \tilde{\mathbf{n}}(\mathbf{y})) \cdot \mathbf{v}(\mathbf{y}) \, ds(\mathbf{y}) = 0,$$

where $\tilde{\mathbf{n}}$ is the unit normal to $\mathbf{u}_V(\partial \mathcal{B}_\delta(\mathbf{x}_0))$. If H is the region of volume V occupied by the cavity induced by \mathbf{u}_V , then the limit above can be replaced by the corresponding integral over ∂H . It follows now that

$$\int_{\partial H} ([\mathbf{T}(\mathbf{y}) + \mu_V \mathbf{I}] \cdot \tilde{\mathbf{n}}(\mathbf{y})) \cdot \mathbf{v}(\mathbf{y}) \, ds(\mathbf{y}) = 0,$$

for all $\mathbf{v} \in C^1(\mathbb{R}^n)$. Thus

$$[\mathbf{T}(\mathbf{y}) + \mu_V \mathbf{I}] \cdot \tilde{\mathbf{n}}(\mathbf{y}) = 0, \text{ over } \partial H, \quad (3.11)$$

in the sense of trace.

4 Numerical results

In this section we describe some of the elements of a numerical procedure, based on the results of the previous sections, to compute a minimizer of (1.6). In addition we work a numerical example in which we check the convergence as $\varepsilon \searrow 0$ predicted by Theorem 3.2.

For given values of ε, V , we use the method outlined in Theorem 2.2 to compute the minimizer \mathbf{u}_ε in (2.3). The minimizers in (2.2a) (dropping the subscript “ j ”) are computed using the *gradient flow equation*²:

$$\Delta \mathbf{u}_t = -\operatorname{div} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}) + (\mu + \eta c_\varepsilon(\mathbf{u}))(\operatorname{adj} \nabla \mathbf{u})^t \right], \text{ in } \Omega_\varepsilon, \quad (4.1)$$

where for all $t \geq 0$, $\mathbf{u}(\mathbf{x}, t) = \mathbf{A}\mathbf{x}$ over $\partial\Omega$ and

$$\left[\nabla \mathbf{u}_t + \frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}) + (\mu + \eta c_\varepsilon(\mathbf{u}))(\operatorname{adj} \nabla \mathbf{u})^t \right] \cdot \mathbf{n} = \mathbf{0}, \text{ on } \partial\mathcal{B}_\varepsilon(\mathbf{x}_0). \quad (4.2)$$

The gradient flow equation leads to a descent method for the solution of (2.2a). (For more details about the gradient flow method and its properties we refer to [16], and for its use in problems leading to cavitation see [8].) After discretization of the partial derivative with respect to “ t ”, one can use a finite element method to solve the resulting flow equation. In particular, if we let $\Delta t > 0$ be given, and set $t_{i+1} = t_i + \Delta t$ where $t_0 = 0$, we can approximate $\mathbf{u}_t(\mathbf{x}, t_i)$ with:

$$\mathbf{z}_i(\mathbf{x}) = \frac{\mathbf{u}_{i+1}(\mathbf{x}) - \mathbf{u}_i(\mathbf{x})}{\Delta t},$$

where $\mathbf{u}_i(\mathbf{x}) = \mathbf{u}(\mathbf{x}, t_i)$, etc.. (We take $\mathbf{u}_0(\mathbf{x})$ to be some initial deformation satisfying the boundary condition on $\partial\Omega$, e.g., $\mathbf{A}\mathbf{x}$.) Inserting this approximation into the weak form of (4.1), (4.2), and evaluating the right hand side of (4.1) at $\mathbf{u} = \mathbf{u}_i$, we arrive at the following iterative formula:

$$\int_{\Omega_\varepsilon} \nabla \mathbf{z}_i : \nabla \mathbf{v} \, d\mathbf{x} + \int_{\Omega_\varepsilon} \left[\frac{dW}{d\mathbf{F}}(\nabla \mathbf{u}_i) + (\mu + \eta c_\varepsilon(\mathbf{u}_i))(\operatorname{adj} \nabla \mathbf{u}_i)^t \right] : \nabla \mathbf{v} \, d\mathbf{x} = 0, \quad (4.3)$$

for all \mathbf{v} vanishing on $\partial\Omega$ and sufficiently smooth so that the integrals above are well defined. Given \mathbf{u}_i , one can solve the above equation for \mathbf{z}_i via some finite element scheme,

²It follows from (2.9) that the Euler–Lagrange equations for (2.2a) are formally given by equating to zero the right hand side of (4.1).

and then set $\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta t \mathbf{z}_i$. This process is repeated for $i = 0, 1, \dots$, until $\mathbf{u}_{i+1} - \mathbf{u}_i$ is “small” enough (10^{-3} in the calculations below), or some maximum value of “ t ” is reached, declaring the last \mathbf{u}_i as an approximation of \mathbf{u}_ε . This whole process is repeated for smaller values of ε , to obtain as a result an approximation of the minimizer \mathbf{u}_V in (1.6).

For the computations we used the stored energy function (1.13) in which:

$$h(d) = c_1 d^{e_1} + c_2 d^{-e_2},$$

where $c_1, c_2 \geq 0$ and $e_1, e_2 > 0$. The reference configuration is stress free provided:

$$c_2 = \frac{\mu(\sqrt{n})^{q-2} + c_1 e_1}{e_2}.$$

The case $\mu = 0$ in (1.13) is called an *elastic fluid*.

For an elastic fluid in which $\Omega = \mathcal{B} \equiv \mathcal{B}_1(\mathbf{0})$ and $\mathbf{x}_0 = \mathbf{0}$, the minimizer \mathbf{u}_V in (1.6) is given by (see [15]):

$$\mathbf{u}_V(\mathbf{x}) = [dR^n + (1-d)]^{1/n} \frac{\mathbf{A}\mathbf{x}}{R}, \quad R = \|\mathbf{x}\|,$$

where d is given by

$$d = 1 - \frac{nV}{\omega_n \det \mathbf{A}}.$$

(V is assumed to be sufficiently small as to guarantee that $d > 0$.) It follows that $\det \nabla \mathbf{u}_V = d \det \mathbf{A}$. Thus we have that

$$E(\mathbf{u}_V) = \int_B h(\det \nabla \mathbf{u}_V) \, d\mathbf{x} = \frac{\omega_n}{n} h(d \det \mathbf{A}),$$

where ω_n denotes the area of the unit sphere in \mathbb{R}^n . We now consider the particular case in which $n = 2$, $c_1 = 1$, $e_1 = 2$, $e_2 = 1$, $V = \pi(0.15)^2$, and $\mathbf{A} = \text{diag}(1.1, 1.4)$. Using the formulas above, we get that

$$E(\mathbf{u}_V) = \pi h((1.1)(1.4) - 0.15^2) = 11.3750.$$

For the parameters in Theorem 2.2 we used $\gamma = 0.25$, $\beta = 2$, with the stopping criteria in (2.2) given by $|\mu_{j+1} - \mu_j| < 10^{-3}|\mu_j|$, and for the solution of the sub-problems (4.3) we used the package `freefem++` (see [5]). We show in Table 1 the results in this case for the method described at the beginning of this section. For each ε we clearly see the penalty-multiplier iterations converging as predicted in Theorem 2.2. Note that the

penalty parameters do not become too large, thus avoiding the ill-conditioning associated with large values of these parameters. As we move down along the table and look at the last computed energy for each ε , we see that these values are approaching the exact energy 11.3750, to within the convergence tolerances in the gradient flow and penalty multiplier iterations and finite element approximation, in accordance with the result in Theorem 3.2.

5 Final Comments

In [15, Proposition 6.1] the authors introduced a regularized penalty method for approximating solutions of (1.6). They anticipated without proof, the convergence of the corresponding regularised minimizers to a solution of (1.6). The result in Theorem 3.2 fills that gap and by using a penalty-multiplier scheme we also obtained information on the Lagrange multiplier corresponding to the integral constraint on the volume of the hole produced in the deformed configuration. Moreover, the use of the penalty-multiplier technique leads to a more stable numerical scheme as compared to a standard penalty method, as in general one achieves convergence to a minimum without having to make the penalty parameter excessively large, which could lead to numerical ill conditioning. We anticipate that the scheme introduced in this paper can be used as part of an efficient numerical scheme for the computation of the volume derivative introduced in [15].

A The integral constraint

In this section of the appendix we give a characterization of the condition on the distributional determinant in (1.4) in terms of an integral constraint.

Proposition A.1. *For any $\mathbf{u} \in \mathcal{A}_{\mathbf{A},V}$, the following integral constraint holds:*

$$\int_{\Omega} \det \nabla \mathbf{u} \, d\mathbf{x} = (\det \mathbf{A}) |\Omega| - V. \quad (\text{A.1})$$

Moreover, for any $\mathbf{u} \in C^2(\Omega \setminus \{\mathbf{x}_0\}) \cap C^1(\overline{\Omega} \setminus \{\mathbf{x}_0\})$ that satisfies (A.1), with $\det \nabla \mathbf{u} > 0$ a.e., $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ on $\partial\Omega$, and \mathbf{u} satisfies INV on Ω , it follows that

$$\text{Det} \nabla \mathbf{u} = \det \nabla \mathbf{u} + V \delta_{\mathbf{x}_0}. \quad (\text{A.2})$$

Proof: The first part of the proposition follows essentially from [14, Lemma 8.1] but we refer to [21] for a proof as well.

ε	j	$c_\varepsilon(\mathbf{u}_j)$	$E_{\varepsilon, \mu_j, \eta_j}(\mathbf{u}_j)$	μ_j	η_j
0.1	0	-0.435566	10.5824	0	5
	1	-0.108628	11.3009	-2.17783	10
	2	-0.0182433	11.3627	-3.26411	10
	3	-0.00175846	11.3637	-3.44654	10
	4	0.00123636	11.3636	-3.46413	10
	5	0.00119778	11.3636	-3.45177	20
	6	5.80566e-05	11.3636	-3.42781	40
0.05	0	-0.408565	10.6179	0	5
	1	-0.114678	11.2988	-2.04282	10
	2	-0.0116487	11.3693	-3.18961	20
	3	0.000242224	11.3699	-3.42258	20
	4	0.00100667	11.3699	-3.41774	20
	5	0.000804023	11.3699	-3.3976	40
	6	-3.24128e-05	11.3699	-3.36544	80
0.025	0	-0.201328	10.8508	0	5
	1	-0.193863	11.149	-1.00664	10
	2	-0.0446432	11.3758	-2.94527	20
	3	0.00475689	11.3697	-3.83813	20
	4	0.015619	11.3684	-3.74299	20
	5	0.00201459	11.3716	-3.43061	40
	6	-9.22278e-05	11.3717	-3.35003	40
	7	-9.20066e-05	11.3717	-3.35372	40
	8	-7.96496e-05	11.3717	-3.3574	80
	9	-5.49799e-06	11.3717	-3.36377	160
0.0125	0	-0.0427461	11.2392	0	5
	1	-0.0916523	11.1388	-0.213731	10
	2	-0.0956048	11.2616	-1.13025	20
	3	-0.0281877	11.3806	-3.04235	40
	4	0.00372948	11.3698	-4.16986	80
	5	0.00570313	11.3705	-3.8715	80
	6	0.000615149	11.3721	-3.41525	160
	7	-0.000212807	11.3721	-3.31683	160
	8	-3.51888e-05	11.3721	-3.35088	320
	9	1.10978e-06	11.3721	-3.36214	320
0.00625	0	0.0374087	11.503	0	5
	1	0.0174181	11.4372	0.187044	10
	2	0.00767417	11.4034	0.361225	20
	3	0.00229717	11.3836	0.514708	40
	4	-0.00539297	11.3545	0.606595	80
	5	-0.0106619	11.3448	0.175157	160
	6	-0.00686916	11.3674	-1.53074	320
	7	0.000534898	11.3721	-3.72887	640
	8	5.6007e-05	11.3723	-3.38654	640
	9	5.51664e-07	11.3723	-3.35069	640

Table 1: Convergence of the regularized penalty–multiplier minimizers for the case of a two dimensional elastic fluid.

For the second part of the proposition, let $\mathbf{u} \in C^2(\Omega \setminus \{\mathbf{x}_0\}) \cap C^1(\overline{\Omega} \setminus \{\mathbf{x}_0\})$ with $\det \nabla \mathbf{u} > 0$ a.e., $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ on $\partial\Omega$, and \mathbf{u} satisfies INV on Ω . Assume that \mathbf{u} satisfy (A.1). For any $\phi \in C_0^\infty(\Omega)$ and $\delta > 0$ sufficiently small,

$$\begin{aligned} \int_{\Omega \setminus \overline{B_\delta(\mathbf{x}_0)}} \nabla \phi \cdot (\text{adj } \nabla \mathbf{u}) \mathbf{u} \, d\mathbf{x} &= \int_{\partial(\Omega \setminus \overline{B_\delta(\mathbf{x}_0)})} \phi ((\text{adj } \nabla \mathbf{u}) \mathbf{u}) \cdot \mathbf{n} \, ds \\ &\quad - \int_{\Omega \setminus \overline{B_\delta(\mathbf{x}_0)}} \phi \operatorname{div} ((\text{adj } \nabla \mathbf{u}) \mathbf{u}) \, d\mathbf{x}, \\ &= -\phi(\mathbf{x}_\delta) \int_{\partial B_\delta(\mathbf{x}_0)} ((\text{adj } \nabla \mathbf{u}) \mathbf{u}) \cdot \mathbf{n} \, ds \\ &\quad - n \int_{\Omega \setminus \overline{B_\delta(\mathbf{x}_0)}} \phi \det \nabla \mathbf{u} \, d\mathbf{x}, \end{aligned}$$

where $\mathbf{x}_\delta \in \partial B_\delta(\mathbf{x}_0)$ and \mathbf{n} in the last boundary integral is the unit outer normal to $\partial B_\delta(\mathbf{x}_0)$. Thus:

$$\begin{aligned} \langle \operatorname{Det} \nabla \mathbf{u}, \phi \rangle &= \lim_{\delta \rightarrow 0} \left[-\frac{1}{n} \int_{\Omega \setminus \overline{B_\delta(\mathbf{x}_0)}} \nabla \phi \cdot (\text{adj } \nabla \mathbf{u}) \mathbf{u} \, d\mathbf{x} \right], \\ &= \phi(\mathbf{x}_0) \lim_{\delta \rightarrow 0} \left[\frac{1}{n} \int_{\partial B_\delta(\mathbf{x}_0)} ((\text{adj } \nabla \mathbf{u}) \mathbf{u}) \cdot \mathbf{n} \, ds \right] \\ &\quad + \int_{\Omega} \phi \det \nabla \mathbf{u} \, d\mathbf{x}. \end{aligned} \tag{A.3}$$

Note that:

$$\begin{aligned} \int_{\Omega \setminus \overline{B_\delta(\mathbf{x}_0)}} \det \nabla \mathbf{u} \, d\mathbf{x} &= \frac{1}{n} \int_{\partial(\Omega \setminus \overline{B_\delta(\mathbf{x}_0)})} ((\text{adj } \nabla \mathbf{u}) \mathbf{u}) \cdot \mathbf{n} \, ds, \\ &= \frac{1}{n} \int_{\partial\Omega} ((\text{adj } \nabla \mathbf{u}) \mathbf{u}) \cdot \mathbf{n} \, ds \\ &\quad - \frac{1}{n} \int_{\partial B_\delta(\mathbf{x}_0)} ((\text{adj } \nabla \mathbf{u}) \mathbf{u}) \cdot \mathbf{n} \, ds. \end{aligned} \tag{A.4}$$

Assuming that (A.1) holds, we get that:

$$\begin{aligned} \lim_{\delta \rightarrow 0} \left[\frac{1}{n} \int_{\partial B_\delta(\mathbf{x}_0)} ((\text{adj } \nabla \mathbf{u}) \mathbf{u}) \cdot \mathbf{n} \, ds \right] &= \frac{1}{n} \int_{\partial\Omega} ((\text{adj } \nabla \mathbf{u}) \mathbf{u}) \cdot \mathbf{n} \, ds - \int_{\Omega} \det \nabla \mathbf{u} \, d\mathbf{x}, \\ &= \frac{1}{n} \int_{\partial\Omega} ((\text{adj } \nabla \mathbf{u}) \mathbf{A}\mathbf{x}) \cdot \mathbf{n} \, ds + V - (\det \mathbf{A}) |\Omega|. \end{aligned}$$

Since $\mathbf{u} \in C^1(\overline{\Omega} \setminus \{\mathbf{x}_0\})$ and $\mathbf{u}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for all $\mathbf{x} \in \partial\Omega$, we have that $\nabla \mathbf{u}(\mathbf{x}) = \mathbf{A}$ for all $\mathbf{x} \in \partial\Omega$. Hence

$$\begin{aligned} \int_{\partial\Omega} ((\text{adj } \nabla \mathbf{u}) \mathbf{A} \mathbf{x}) \cdot \mathbf{n} \, ds &= \int_{\partial\Omega} ((\text{adj } \mathbf{A}) \mathbf{A} \mathbf{x}) \cdot \mathbf{n} \, ds, \\ &= \det \mathbf{A} \int_{\partial\Omega} \mathbf{x} \cdot \mathbf{n} \, ds = n (\det \mathbf{A}) |\Omega|. \end{aligned} \quad (\text{A.5})$$

Using this above we get that

$$\lim_{\delta \rightarrow 0} \left[\frac{1}{n} \int_{\partial B_\delta(\mathbf{x}_0)} ((\text{adj } \nabla \mathbf{u}) \mathbf{u}) \cdot \mathbf{n} \, ds \right] = V.$$

Thus combining this with (A.3) we get that:

$$\langle \text{Det } \nabla \mathbf{u}, \phi \rangle = \int_{\Omega} \phi \det \nabla \mathbf{u} \, d\mathbf{x} + V \phi(\mathbf{x}_0), \quad \forall \phi \in C_0^\infty(\Omega),$$

i.e., that (A.2) holds. □

B The constrained admissible set

We now show that for ε sufficiently small, the admissible sets in (1.8) are non-empty. In the proof we make use of the following result. let $\mathcal{B}_1(\mathbf{0})$ be the unit ball with centre at the origin and for any $d \in (0, 1)$, define

$$\mathbf{u}_d(\mathbf{x}) = [dR^n + (1 - d)]^{1/n} \frac{\mathbf{x}}{R}, \quad R = \|\mathbf{x}\|, \quad \mathbf{x} \in \mathcal{B}_1(\mathbf{0}). \quad (\text{B.1})$$

It follows from [2, Lemma 4.1], that $\mathbf{u}_d \in W^{1,p}(\mathcal{B}_1(\mathbf{0}))$ for $p \in [1, n)$. An easy calculation shows as well that $\det \nabla \mathbf{u}_d = d$.

Lemma B.1. *There exists $V_0, \varepsilon_0 > 0$ such that*

$$\mathcal{C}_{\mathbf{A}}^\varepsilon \equiv \{\mathbf{u} \in \mathcal{A}_{\mathbf{A}}^\varepsilon \mid c_\varepsilon(\mathbf{u}) = 0\} \neq \emptyset,$$

for all $\varepsilon \in [0, \varepsilon_0)$ and $0 < V < V_0$. Moreover, if W is nonnegative and for any $0 < \gamma < \delta$ there exists a constant $K > 0$ such that

$$W(\mathbf{F}) \leq K(\|\mathbf{F}\|^p + 1), \quad \text{whenever } \det \mathbf{F} \in [\gamma, \delta], \quad (\text{B.2})$$

then for any nonnegative sequence $\varepsilon_j \rightarrow 0$, there exists a sequence $\mathbf{z}_{\varepsilon_j} \in \mathcal{C}_{\mathbf{A}}^{\varepsilon_j}$ such that

$$E_{\varepsilon_j}(\mathbf{z}_{\varepsilon_j}) \leq C, \quad \forall j,$$

for some constant $C > 0$.

Proof: Let $\eta > 0$ be such that $\mathcal{B}_\eta(\mathbf{x}_0) \subset \Omega$ and define

$$V_0 = \eta^n \frac{\omega_n}{n} \det \mathbf{A}, \quad \varepsilon_0^n = \frac{nV}{\omega_n \det \mathbf{A}}.$$

For $0 < V < V_0$ and $\varepsilon \in [0, \varepsilon_0)$, let

$$d_\varepsilon = \frac{1}{\eta^n - \varepsilon^n} \left(\eta^n - \frac{nV}{\omega_n \det \mathbf{A}} \right).$$

It follows now that $d_\varepsilon \in (0, 1)$ and that $B_\varepsilon(\mathbf{x}_0) \subset \mathcal{B}_\eta(\mathbf{x}_0)$. Now define $\mathbf{w}_\varepsilon = \mathbf{u}_{d_\varepsilon}$ and let

$$\mathbf{v}_\varepsilon(\mathbf{x}) = \begin{cases} \eta \mathbf{A} \mathbf{w}_\varepsilon \left(\frac{\mathbf{x} - \mathbf{x}_0}{\eta} + \mathbf{x}_0 \right) + \mathbf{A} \mathbf{x}_0 - \eta \mathbf{A} \mathbf{x}_0 & , \quad \mathbf{x} \in \mathcal{B}_\eta(\mathbf{x}_0), \\ \mathbf{A} \mathbf{x} & , \quad \mathbf{x} \in \Omega \setminus \mathcal{B}_\eta(\mathbf{x}_0). \end{cases}$$

Clearly $\mathbf{v}_\varepsilon(\mathbf{x}) = \mathbf{A} \mathbf{x}$ on $\partial\Omega \cup \partial\mathcal{B}_\eta(\mathbf{x}_0)$. Since $\mathbf{w}_\varepsilon \in W^{1,p}(\mathcal{B}_1(\mathbf{0}))$ for $p \in [1, n)$, it follows that $\mathbf{v}_\varepsilon \in W^{1,p}(\Omega)$ for $p \in [1, n)$. Hence in particular, if $\mathbf{z}_\varepsilon = \mathbf{v}_\varepsilon|_{\Omega_\varepsilon}$, then $\mathbf{z}_\varepsilon \in \mathcal{A}_{\mathbf{A}}^\varepsilon$ for $\varepsilon \geq 0$. That $\mathbf{z}_0 \in \mathcal{A}_{\mathbf{A}}^0$ follows from the following result:

$$\text{Det} \nabla \mathbf{w}_0 = \det \nabla \mathbf{w}_0 + \frac{\omega_n}{n} (1 - d_0) \delta_0,$$

where δ_0 is the Dirac delta measure at the origin. An easy calculation now shows that

$$\det \nabla \mathbf{z}_\varepsilon = \begin{cases} d_\varepsilon \det \mathbf{A} & , \quad \mathbf{x} \in \mathcal{B}_\eta(\mathbf{x}_0) \setminus \mathcal{B}_\varepsilon(\mathbf{x}_0), \\ \det \mathbf{A} & , \quad \mathbf{x} \in \Omega \setminus \mathcal{B}_\eta(\mathbf{x}_0). \end{cases}$$

Using the definition of d_ε , we get now that

$$\begin{aligned} \int_{\Omega_\varepsilon} \det \nabla \mathbf{z}_\varepsilon \, d\mathbf{x} &= \int_{\mathcal{B}_\eta(\mathbf{x}_0) \setminus \mathcal{B}_\varepsilon(\mathbf{x}_0)} \det \nabla \mathbf{z}_\varepsilon \, d\mathbf{x} + \int_{\Omega \setminus \mathcal{B}_\eta(\mathbf{x}_0)} \det \nabla \mathbf{z}_\varepsilon \, d\mathbf{x}, \\ &= (\det \mathbf{A}) \frac{\omega_n}{n} d_\varepsilon (\eta^n - \varepsilon^n) + \det \mathbf{A} \left(|\Omega| - \frac{\omega_n}{n} \eta^n \right), \\ &= (\det \mathbf{A}) |\Omega| - V, \end{aligned}$$

that is, $c_\varepsilon(\mathbf{z}_\varepsilon) = 0$. Hence $\mathbf{z}_\varepsilon \in \mathcal{C}_{\mathbf{A}}^\varepsilon$.

For the second part of the lemma, we observe that for any nonnegative sequence (ε_j) converging to zero, we can conclude from [2, Eqn. (4.5)] that $(\mathbf{v}_{\varepsilon_j})$ is a Cauchy sequence in $W^{1,p}(\Omega)$. Since $\mathbf{v}_{\varepsilon_j} \rightarrow \mathbf{v}_0$ a.e., we have that $\mathbf{v}_{\varepsilon_j} \rightarrow \mathbf{v}_0$ in $W^{1,p}(\Omega)$. From (B.2) and since $(\det \nabla \mathbf{v}_{\varepsilon_j})$ converges a.e. to

$$\begin{cases} d_0 \det \mathbf{A} & , \quad \mathbf{x} \in \mathcal{B}_\eta(\mathbf{x}_0), \\ \det \mathbf{A} & , \quad \mathbf{x} \in \Omega \setminus \mathcal{B}_\eta(\mathbf{x}_0). \end{cases}$$

we get that

$$E_{\varepsilon_j}(\mathbf{z}_{\varepsilon_j}) = \int_{\Omega_{\varepsilon_j}} W(\nabla \mathbf{z}_{\varepsilon_j}) \, d\mathbf{x} \leq \int_{\Omega} W(\nabla \mathbf{v}_{\varepsilon_j}) \, d\mathbf{x} \leq K(\|\nabla \mathbf{v}_{\varepsilon_j}\|^p + 1) \leq C,$$

for some constant $C > 0$. □

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